

# QCD Phase Transitions in the $1/N_c$ Expansion

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We study the QCD phase diagram at nonzero baryon and isospin chemical potentials using the  $1/N_c$  expansion. We find that there are two phase transitions between the hadronic phase and the quark gluon plasma phase. We discuss the consequences of this result for the universality class of the critical endpoint at nonzero baryon and zero isospin chemical potential.

## I. INTRODUCTION

In order to understand neutron stars, the early Universe, and heavy ion collision experiments, it is necessary to better grasp the physics of strong interacting matter in extreme conditions. Therefore the study of the QCD phase diagram at nonzero temperature and densities is very important. The nonperturbative lattice simulations that successfully addressed problems at zero densities can be used at nonzero isospin density [1], but not at nonzero baryon density because of the so-called "sign problem". One has therefore to rely on novel approaches to study QCD at nonzero baryon density [2, 3, 4, 5], which corresponds to the most important physical situations. In particular, these new methods have been used to study the critical temperature that separates the hadronic phase from the quark gluon plasma phase. However, these studies are valid only at small chemical potentials.

In a previous article, we have successfully used the  $1/N_c$  expansion of QCD to explain some key properties of the critical temperature that separates the hadronic phase from the quark gluon plasma phase [6]. In the present work we shall extend our study to the general case  $\mu_u \neq \mu_d$  and investigate the chiral phase transitions. Several models have shown that the phase diagram might be qualitatively altered in this case [7, 8, 9]. We shall also describe the consequences of our results for the universality class of the critical endpoint. We shall restrict ourselves to phase transitions between the hadronic phase and the quark gluon plasma phase, i.e. to situations with  $\mu_I < m_\pi$ , avoiding conditions where the ground state becomes a pion superfluid at  $T = 0$  [10, 11, 12, 13].

## II. CRITICAL TEMPERATURES

We use the usual  $1/N_c$  expansion with 't Hooft's coupling [14]. As explained in detail in [6, 15], the  $1/N_c$  expansion of the pressure reads:

$$p(T, \{m_f, \mu_f\}) = N_c^2 \left( p_0(T) + \frac{1}{N_c} \sum_{f=1}^{N_f} p_1(T, m_f, \mu_f^2) + \mathcal{O}\left(\frac{1}{N_c^2}\right) \right), \quad (1)$$

where  $f$  are the different light quark flavors. The sets of diagrams that contribute to the pressure are shown in Fig. 1. In a finite volume, the specific heat will peak at the transition between the hadronic phase and the quark gluon

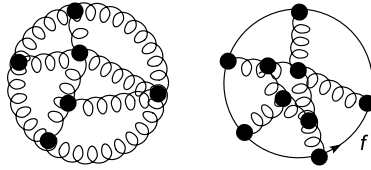


FIG. 1: Sets of diagrams that contribute to the pressure. The symbol  $f$  denote the quark flavor and runs over all flavors. The first diagram is  $\mathcal{O}(N_c^2)$ , and the second is  $\mathcal{O}(N_c)$ .

plasma phase. This peak might diverge in the thermodynamic limit, depending on whether there is a genuine phase transition or only a crossover. The specific heat can be derived directly from the pressure leading to [6]

$$C_V = N_c^2 \left( c_0(T) + \frac{1}{N_c} \sum_{f=1}^{N_f} c_1(T, m_f, \mu_f^2) + \mathcal{O}\left(\frac{1}{N_c^2}\right) \right). \quad (2)$$

Therefore, the critical temperature, which can be obtained by solving  $\partial C_V / \partial T|_{T_c} = 0$  is given by [6]:

$$t_c = \frac{1}{N_c} \sum_{f=1}^{N_f} f(\mu_f^2) + \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (3)$$

where the reduced temperature  $t_c = (T_c - T_0)/T_0$ , with  $T_0$  the critical temperature at zero chemical potentials.

The chiral susceptibilities,  $\chi_f = \partial^2 p / \partial m_f^2$ , peak at the transition or crossover where chiral symmetry is partially restored (chiral symmetry cannot be completely restored since we restrict ourselves to the case  $m_f \neq 0$ ). The diagrams that contribute to  $\chi_f$  up to next-to-next-to-leading order are shown in Fig. 2. The  $1/N_c$  expansion of the

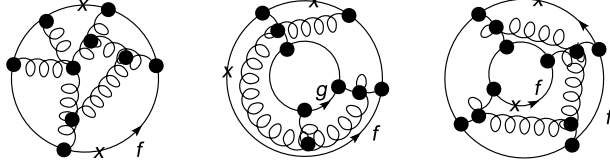


FIG. 2: Sets of diagrams that contribute to the chiral susceptibility  $\chi_f$ . The crosses are insertions of the mass operator,  $f$  is the specific quark flavor associated with  $\chi_f$ , and  $g$  runs over all quark flavors. The first diagram is  $\mathcal{O}(N_c)$ , the second and third are  $\mathcal{O}(N_c^0)$ .

chiral susceptibilities can be expressed as

$$\chi_f = N_c \left( \chi_0(T, \mu_f^2) + \frac{1}{N_c} \left( N_f \chi_1(T, \mu_f^2) + \sum_{g=1}^{N_f} (\chi_2(\mu_g^2) + \chi_3(\mu_f \mu_g)) + \chi_4(\mu_f^2) \right) + \mathcal{O}\left(\frac{1}{N_c^2}\right) \right), \quad (4)$$

where  $\chi_0$  comes from the first diagram in Fig. 2,  $\chi_{1,2,3}$  come from the second diagram in Fig. 2 with the  $\mu$ -dependence taken respectively from the outside quark loop only ( $\chi_1$ ), the inside quark loop only ( $\chi_2$ ), and both the inside and outside quark loops ( $\chi_3$ ), and  $\chi_4$  comes from the third diagram in Fig. 2. In the equation above, we have used that  $p(\mu_u, \mu_d) = p(-\mu_u, -\mu_d)$  because of CP, and thus that  $\chi_f(\mu_u, \mu_d) = \chi_f(-\mu_u, -\mu_d)$ , and that for equal masses,  $\chi_u(\mu_u, \mu_d) = \chi_d(\mu_d, \mu_u)$ . If we restrict ourselves to  $m_u = m_d$ , notice that the that these latter two properties imply that  $\chi_u = \chi_d$  for  $\mu_u = \pm \mu_d$ , but that  $\chi_u \neq \chi_d$  in general since  $\mu_u^2 \neq \mu_d^2$ . Therefore, since the critical temperature for the restoration of chiral symmetry is defined by a peak in  $\chi_f$ , each flavor might have a different critical temperature when  $\mu_u^2 \neq \mu_d^2$ .

The critical temperature that corresponds to the restoration of chiral symmetry,  $T_f$ , can be obtained from  $\partial \chi_f / \partial T|_{T_f} = 0$ . Therefore, we find that the reduced temperature for chiral symmetry restoration,  $t_f = (T_f - T_0)/T_0$ , is given by

$$t_f = \tau_0(\mu_f^2) + \frac{1}{N_c} \left( N_f \tau_1(\mu_f^2) + \sum_{g=1}^{N_f} (\tau_2(\mu_g^2) + \tau_3(\mu_f \mu_g)) + \tau_4(\mu_f^2) \right) + \mathcal{O}\left(\frac{1}{N_c^2}\right). \quad (5)$$

Lattice simulations have shown that there is only one phase transition when  $\mu_u = \pm \mu_d$ , at least at small chemical potentials for  $N_c = 2$  and 3 [1, 3, 4, 16]. We assume that this property holds at large  $N_c$ . Therefore the critical temperatures  $t_c$  and  $t_f$  for each flavor should coincide, at least at small chemical potential. However, it is clear from (3) and (5) that  $t_c$  and  $t_f$  do not have the same  $1/N_c$  expansion: the chemical potential enters at a different order in the  $1/N_c$  expansion, and, at leading order,  $t_c$  depends equally on all chemical potentials, whereas  $t_f$  depends only on one chemical potential. As was shown in [6], the qualitative properties of  $t_c$  are in agreement with the lattice results. We therefore impose that  $t_c = t_u = t_d$  for  $\mu_u = \pm \mu_d$  as a function of  $N_f$  and  $N_c$  in the  $1/N_c$  expansion. These constraints imply that

$$\begin{aligned} \tau_0(\mu^2) &= 0 \\ \tau_3(\mu^2) &= \tau_3(-\mu^2) \\ \tau_4(\mu^2) &= 0 \\ f(\mu^2) &= \tau_1(\mu^2) + \tau_2(\mu^2) + \tau_3(\mu^2). \end{aligned} \quad (6)$$

Notice that the same conclusions for QCD with an even number of flavors can be reached by using the dependence of  $t_c$  and  $t_f$  on the quark masses rather than on the chemical potentials.

In general, we expect that  $\tau_{1,3} \neq 0$ . We therefore conclude that critical temperatures for two different flavors will be related by

$$t_f - t_g = \frac{1}{N_c} \left( N_f (\tau_1(\mu_f^2) - \tau_1(\mu_g^2)) + \sum_{h=1}^{N_f} (\tau_3(\mu_f \mu_h) - \tau_3(\mu_g \mu_h)) \right) + \mathcal{O}\left(\frac{1}{N_c^2}\right), \quad (7)$$

and that the critical temperatures are sensitive to *all* chemical potentials at the *same* order in the  $1/N_c$  expansion. Therefore the restoration of the chiral symmetry takes place at different temperatures at  $\mu_u^2 \neq \mu_d^2$ , i.e. at nonzero baryon *and* isospin chemical potentials. This is important since most experiments are precisely done in these conditions. A similar relation between the critical temperatures for different flavors can also be written as a function of the quark masses at zero chemical potentials.

### III. PHASE DIAGRAM

We shall now analyze the consequences of this  $1/N_c$  analysis for the QCD phase diagram. We shall restrict ourselves to small positive chemical potentials, i.e.  $\mu_u, \mu_d > 0$ . Possible generic phase diagrams resulting from the above analysis are sketched in Fig. 3.

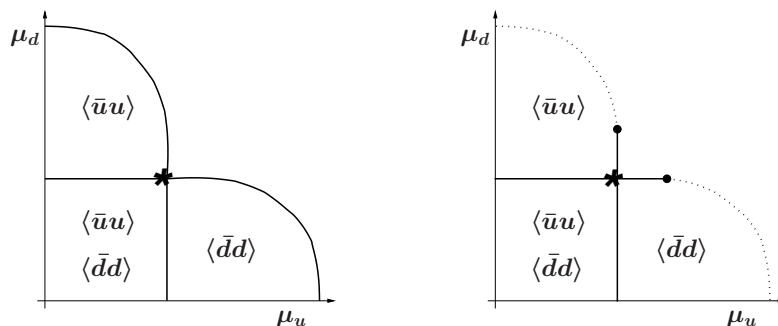


FIG. 3: Generic phase diagrams at fixed  $T$  in the  $(\mu_u, \mu_d)$  plane in the  $1/N_c$  expansion. The chiral condensates are indicated where they are large. The solid curves are first order phase transitions, and the dotted curves are crossovers. The dots are critical endpoints, and the stars denote the points that become tetracritical endpoints at some temperature. See text for more details.

In the phase diagrams presented in Fig. 3, the temperature is high enough so that the pion condensation phase is absent for  $\mu_u, \mu_d > 0$  (or occupies at most a small portion of the phase diagrams near the  $\mu_u$  and  $\mu_d$  axes that we will ignore here). The temperature in these diagrams is below the temperature of the critical endpoint at  $\mu_u = \mu_d$ ,  $T_e$  [17, 18, 19]. Above this temperature, all the transitions become crossovers. The points denoted by stars in the phase diagrams of Fig. 3 become tetracritical endpoints at  $T = T_e$ . The universality class of these points is that of the  $Z_2 \times Z_2$  Ising model, since there are two massless sigma modes that correspond to the divergence of both the baryon and isospin susceptibilities.

It is possible that accidental circumstances might alter the generic phase diagrams presented in Fig. 2. If  $\tau_1 = 0$  and  $\tau_3 = 0$ , the chiral phase transitions would take place at the same temperature and chemical potentials for all flavors. Another accident could lead to the merging of the two critical endpoints in the second phase diagram of Fig. 3 into one single point at the corner of the region where all chiral condensates are large. In this case, the critical endpoint at  $T_e$  would be in the universality class of the  $Z_2$  Ising model, with only one massless sigma mode. However, we consider these latter two phase diagrams to be accidental in the  $1/N_c$  expansion: They are accidents in the same way as  $\tau_0 = 0$  is an accident.

Similar phase diagrams as those presented in Fig. 3 can also be obtained from a slightly modified version of a Random Matrix model [7, 19]. In the original model, at sufficiently high temperatures, the pion condensation phase never appears for  $\mu_u, \mu_d > 0$  [7, 8, 9] and the flavor sectors decouple from each other. We use the Random Matrix effective potential and artificially add flavor-mixing terms to it:

$$\Omega_{RMT} = (\sigma_u - m)^2 + (\sigma_d - m)^2 - \frac{1}{4} \log \left[ (((\sigma_u^2 - (\mu_u^2 + a\mu_d^2) + T^2)^2 + 4T^2\mu_u^2)((\sigma_d^2 - (\mu_d^2 + a\mu_u^2) + T^2)^2 + 4T^2\mu_d^2) + b(\mu_u^2 - \mu_d^2)^2\sigma_u^2\sigma_d^2) \right], \quad (8)$$

where  $\sigma_{u,d}$  represent the chiral condensates. Depending on the choice of the artificially introduced parameters  $a$  and  $b$ , one can obtain either of the phase diagrams presented in Fig. 3.

## IV. CONCLUSIONS

In this article we have used the  $1/N_c$  expansion to study the QCD phase diagram at nonzero baryon and isospin chemical potentials, which corresponds to the most common experimental conditions. We have limited our study to the phase transitions from the hadronic phase to the quark-gluon plasma phase. We have also used constraints derived from lattice simulations, in particular that the chiral and deconfinement phase transitions take place at the same temperature for small baryon and isospin chemical potentials. For QCD with two flavors, we have found that there are two chiral phase transitions in general. The QCD phase diagram at nonzero baryon chemical potential is thus qualitatively different at zero and nonzero isospin chemical potential, as other models have predicted [7, 8, 9]. This is important since most experiments are done at nonzero baryon and isospin chemical potentials. As a consequence, the universality class of the critical endpoint at nonzero baryon and zero isospin chemical potentials should be that of the  $Z_2 \times Z_2$  Ising model. The nature of the critical endpoint has observable consequences for heavy ion collision experiments [21]. We conjecture that the real phase diagram resembles the second one in Fig. 3, because, following the analysis of Pisarski and Wilczek [20], the restoration of chiral symmetry for one flavor should be a crossover. The question of the deconfinement phase transition remains open. However, we conjecture that it will take place where the first chiral transition happens, since it is widely believed that the deconfinement phase transition cannot happen after a chiral phase transition.

Finally, we want to comment on the possibility to test the predictions made above. First the properties of the sets of diagrams that enter into the chiral susceptibility and in particular the relations (6) can be tested on the large- $N_c$  lattice approach outlined in [22]. We stress that these tests can also be carried at zero chemical potential by studying these diagrams as a function of the quark masses. Second, the current lattice techniques used for  $\mu_u = \mu_d$  can easily be extended to the more general case  $\mu_u \neq \mu_d$ . It might however be difficult to see the two chiral phase transitions on the lattice in this case, since these studies are limited to small chemical potentials and that the separation between the critical temperatures should be proportional to  $\mu_B \mu_I / N_c$ . Third, the  $1/N_c$  expansion also leads to different critical temperatures for the different flavors if they have different masses. Recent lattice results at zero chemical potentials have shown that the critical temperatures indeed differ for two light quarks and one heavier quark [23], in agreement with the  $1/N_c$  expansion analysis presented here.

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- [1] J. B. Kogut and D. K. Sinclair, Phys. Rev. D **66**, 034505 (2002); Phys. Rev. D **70**, 094501 (2004); arXiv:hep-lat/0509095; D. K. Sinclair, J. B. Kogut and D. Toublan, Prog. Theor. Phys. Suppl. **153**, 40 (2004).
  - [2] Z. Fodor and S. D. Katz, Phys. Lett. B **534**, 87 (2002); JHEP **0203**, 014 (2002); JHEP **0404**, 050 (2004);
  - [3] C. R. Allton *et al.*, Phys. Rev. D **66**, 074507 (2002); Phys. Rev. D **68**, 014507 (2003); F. Karsch *et al.*, Nucl. Phys. Proc. Suppl. **129**, 614 (2004); C. R. Allton *et al.*, Phys. Rev. D **71**, 054508 (2005).
  - [4] P. de Forcrand and O. Philipsen, Nucl. Phys. B **642**, 290 (2002); Nucl. Phys. B **673** (2003) 170; Nucl. Phys. Proc. Suppl. **129**, 521 (2004).
  - [5] M. D'Elia and M. P. Lombardo, Phys. Rev. D **67**, 014505 (2003).
  - [6] D. Toublan, Phys. Lett. B **621**, 145 (2005).
  - [7] B. Klein, D. Toublan and J. J. M. Verbaarschot, Phys. Rev. D **68**, 014009 (2003); Phys. Rev. D **72**, 015007 (2005).
  - [8] D. Toublan and J. B. Kogut, Phys. Lett. B **564**, 212 (2003); M. Frank, M. Buballa and M. Oertel, Phys. Lett. B **562**, 221 (2003); A. Barducci, R. Casalbuoni, G. Pettini and L. Ravagli, Phys. Rev. D **69**, 096004 (2004).
  - [9] A. Barducci, G. Pettini, L. Ravagli and R. Casalbuoni, Phys. Lett. B **564**, 217 (2003).
  - [10] J. B. Kogut, M. A. Stephanov and D. Toublan, Phys. Lett. B **464**, 183 (1999); J. B. Kogut *et al.*, Nucl. Phys. B **582**, 477 (2000);
  - [11] D. T. Son and M. A. Stephanov, Phys. Rev. Lett. **86**, 592 (2001).
  - [12] J. B. Kogut and D. Toublan, Phys. Rev. D **64**, 034007 (2001).
  - [13] K. Splittorff, D. Toublan and J. J. M. Verbaarschot, Nucl. Phys. B **620**, 290 (2002); Nucl. Phys. B **639**, 524 (2002).
  - [14] G. 't Hooft, Nucl. Phys. B **72**, 461 (1974).
  - [15] T. D. Cohen, Phys. Rev. D **70**, 116009 (2004).
  - [16] J. B. Kogut, D. Toublan and D. K. Sinclair, Phys. Lett. B **514**, 77 (2001); Nucl. Phys. B **642**, 181 (2002).
  - [17] A. Barducci *et al.*, Phys. Rev. D **41**, 1610 (1990); A. Barducci *et al.*, Phys. Rev. D **49**, 426 (1994).

- [18] J. Berges and K. Rajagopal, Nucl. Phys. B **538**, 215 (1999).
- [19] M. A. Halasz, A. D. Jackson, R. E. Shrock, M. A. Stephanov and J. J. M. Verbaarschot, Phys. Rev. D **58**, 096007 (1998).
- [20] R. D. Pisarski and F. Wilczek, Phys. Rev. D **29**, 338 (1984).
- [21] M. A. Stephanov, K. Rajagopal and E. V. Shuryak, Phys. Rev. Lett. **81**, 4816 (1998); Phys. Rev. D **60**, 114028 (1999).
- [22] R. Narayanan and H. Neuberger, Phys. Rev. Lett. **91**, 081601 (2003).
- [23] C. Bernard *et al.* [MILC Collaboration], Phys. Rev. D **71**, 034504 (2005) [arXiv:hep-lat/0405029].